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PUZZLING MATHEMATICAL PROBLEMS

for a secondary school student including a theoretical supplement

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PREFACE

This book is not typically aimed at a specific secondary school class. Rather than that it is for a discerning student no matter the year of secondary education. This book is also suitable for a Maths teacher whose objective is to avoid monotony in their class and to attract its attention.

In order to do most of the tasks you need not perfectly master the theory. What is more, the tasks are not supposed to check theoretical knowledge. Some wit, and open mind - the so called "mathematical culture" are useful to do the tasks. The necessary theory, which is not in the syllabus, will be explained by the way. The best is to check the solutions after a successful (or unsuccessful) personal attempts. Certainly, all the problems are solved and explained. Their difficulty is differentiated. The book includes quite simple exercises as well as complicated ones, with the difficulty level of examinations or school contests (The Maths Olympiad). The more complex problems have not been specially distinguished. Rather than that, on purpose, they have been mixed with the easier ones.

The author has made every effort to introduce complete and quite simple solutions. However, if the Reader finds a different, perhaps a more straightforward way to solve a problem, the author will be extremely grateful for sharing this knowledge with him. Undoubtedly, this solution will be published in the next edition of the book. You are welcome to compete and suggest your ideas in this way and send all your comments on:

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Mariusz Kawecki

Mathematical problems

1. **Logic in practice**.

Logic is the basis for every rational activity that a human being takes. Logic in Mathematics does not only involve checking the formula with 0-1 method, but most of all, it involves the ability to use the formulae, draw conclusions and introduce such a modification which will be the most suitable at that moment. The set of exercises below will make you have a little "warm-up" before the problems that are going to appear later. Precision and accurate thinking, which logic is all about, are more important here than just the fact of knowing the formula.

Problem 1

Let us assume that the person we are talking with is truthful, and his understanding of Mathematics is quite good. Yet, the only answer that he can give is "yes" or "no". How many questions do we have to ask him to find out about his phone number? Let us assume it is a five-digit number.

Problem 2

Imagine a visitor coming to a hotel. This person has no money, however, he does possess a gold chain made of 7 links. The owner of the hotel has made an agreement with the visitor to pay with one link of the chain for each day of his stay. However, the payment is supposed to be done systematically (without advances or delays). Only one link can be cut in the chain. Which link should be cut, and how to make the payment to fulfil the agreement?

Problem 3

Imagine two stakes, both 1 m high, put in the ground. Between them there is a string, which is also 1 m long, fixed at the top of the stakes. The string is loose, and its lowest point is 0,5 m over the ground. What is the distance between the stakes?

Problem 4

There are sets of two balls each in three non-transparent urns. In one of the urns there are two white balls, in another two black balls, and in the third one there is a black ball and a white one. Each urn has been labelled with its appropriate name e.g. "white-white" which marks the urn with the white balls. Accidentally all the labels have been mistaken, and as a result, none of them has the correct label. How many balls have to be taken out, and out of which container must they be taken out in order to get to know what are the correct urn labels?

Problem 5

Two ferries go with a constant speed from the opposite banks of a river. The first time they pass each other 90 m. off one of the sides. Each of the ferries reaches the opposite side, takes an immediate turn and comes back. The second time they pass each other 50 m. off the other side. What is the width of the river?

Problem 6
There was a small pond in which only one species of fish, crucian (carps), lived. A fisherman put twenty roaches into the pond. The next day the fisherman cast a net and caught 12 carps and 4 roaches. Can we estimate how many carps lived in the pond?

Problem 7
Imagine a train going at the speed of 60 km/h. The driver of the train lets a dove out 120 km ahead of the station. The dove, flying 80 km/h, reaches the station, turns around and heads back for the train. Then, it reaches the train, turns around again and makes for the station. The bird does both ways until the train arrives at the station. What distance does the dove cover?

Problem 8
Three boys collected $3 (each of them gave $1) and asked another boy to buy a carton of juice. The juice cost $2,5. The boy buying the juice reasoned: there is going to be 50 cents change, and it is undividable by 3, so I will give them the juice and 10 cents change each. Thus, there is going to be 20 cents change for me. Then, it seems that overall the boys collected 90 cents each (each of them got a 10-cent change), so they spent $2.70. The boy who had bought the juice got 20 cents, which gives $2.90 altogether. What happened to the remaining 10 cents (out of $3 the boys had collected)?

Problem 9
There is a number which ends with 2 and it is characterised by the quality that if the last digit of the number is moved to the beginning, a number, which is twice as big will be achieved. What is the number?

Problem 10
Let us make two operations in which most of the numbers have been changed into asterisks.

```
a)        * 1 *              b)        * * 5
        x  3 * 2                     x  1 * *
       -----------                   ----------
           * 3 *                       2 * * 5
       3 * 2 *                       1 3 * 0
  +  * 2 * 5                      +  * * *
  -------------------              ----------------
     1 * 8 * 3 0                     4 * 7 7 *
```

Problem 11
There are two jugs with liquids. In one of them there are 7 glasses of water, and in the other there are 7 glasses of juice. One glass of water is poured into the juice jug and stirred. Then, one glass of the mixture is poured into the water jug. Which is there more of: the juice in the water or the water in the juice?

Problem 12
The police have detained three men X, Y, and Z suspected of a theft. During an interrogation, the men testified as follows:

X: I didn't steal it. Y didn't steal it.
Y: X didn't steal it. Z stole it.
Z: I didn't steal it. X stole it.

It has been determined that one of the men has lied twice, another has told the truth twice, and another has lied once and also told the truth once. Which of the men committed the crime and who lied and told the truth?

Problem 13
There are ten bags with coins. In nine of them there are real coins weighing 10 g each. In one of the bags there are fake coins which weigh 11 g each. How to find out which bag contains the fake coins with one weighing only?

Problem 14
A sultan had one hundred magicians. However, he suspected that they were frauds. So he made up a test for them. He ordered the magicians to stand in a straight line in the way that each magician could see the ones standing ahead but could not see those standing behind him. Each magician had to wear a white or a black cap without knowing what colour he was wearing. However, each of them could see the colour of the caps worn by the magicians standing ahead. The sultan's executioner had to ask each magician about the colour of his cap beginning from the last in the line. They could answer only "black" or "white". The first magician who was supposed to answer incorrectly was lucky to have his life saved. Every other magician making a mistake, however, was supposed to be beheaded. After a discussion the magicians managed to outsmart the sultan. None of them was killed. How did they do it?

Problem 15
Imagine a tape put around an orange along its circumference. The closely sticking tape is then cut and its length is increased by 1 cm. It is obvious that after this operation the tape will not tightly stick to the orange but it will be loosen. Let us assume that a tape is also tightly put around the earth's equator (assume that the earth is a perfect sphere). The tape is also cut and its length is also increased by 1 cm. Which distance will be bigger: the tape to the orange or the tape to the earth?

Problem 16
The following two tasks belong to a group of "inheritance division" ones. The first is an old Arab puzzle, and the other a Roman puzzle seriously considered by an outstanding ancient lawyer Salvian Julian.

a) An Arab man died leaving the following last will:
The oldest son will inherit half of my camels, the middle son will inherit a third of the camels, and the youngest son will inherit a ninth of them.

As the Arab man had 17 camels, his sons had a problem how to divide the inheritance. However, a judge (wise qadi) going nearby on his camel managed to solve the problem. How did he do it?

b) A Roman citizen died widowing his pregnant wife and leaving the following will:
If a baby boy is born, he will inherit 2/3 of my property, and my wife will inherit 1/3 of it. If, on the other hand, a baby girl is born, she will come into 1/3 of my property, and my wife 2/3 of it.

The Roman's wife delivered twins so how was the inheritance supposed to be divided?

Problem 17
There is a number of teams taking part in a competition, in which all teams are supposed to play with one another. The matches can only be won or lost, without draws. Prove that if two teams have got the same number of matches won, there are three teams A, B, C , and A has won with B, B has won with C and C has won with A.

Problem 18
There are 100 balls of different colours in a non-transparent urn: 28 red balls, 20 white balls, 12 black balls, 20 green balls, 10 blue balls, 10 yellow ones. What is the smallest number of balls that must be taken out to be certain that there will be 15 balls of the same colour?

Problem 19
Thirty pikes (a kind of fish) were put into a pond. The pikes ate up one another. A pike feels full after it has eaten at least 3 another pikes. What is the maximum number of full pikes in the pond?

Problem 20
In the equation $101 - 102 = 1$ you are supposed to move one digit only to make it true.

Problem 21
How to cut a round pie with three straight cuts in order to get the following number of pieces:

a) 4, b) 5, c) 6, d) 7?

Problem 22
There are five coins on the table. The one in the middle is put with the head up. The others are put with the tail up. Three coins can be put reversely with one move. How many and what kind of moves are needed to make all the coins lie with their heads up?

Problem 23
Imagine an exotic island inhabited by two tribes. These are: the Zulugulan tribe whose members always tell the truth, and the Guluzulan tribe whose members always lie. There was a traveller, who had hired a native to give him a guide around

the island. On their way they met a man and the traveller wanted the guide to ask the man if he belonged to the Zulugulan tribe. The guide did so and answered the traveller that the man had confirmed being a member of the Zulugulan tribe. Establish which tribe the guide belonged to.

Problem 24

Ann is waiting for a bus to come. Which of the following is the most probable:
A – Ann has been waiting for the bus not less than one minute.
B – Ann has been waiting for the bus not less than two minutes.
C – Ann has been waiting for the bus not less than five minutes.

Problem 25

A 3-digit number ABB has the following property. The product of its digits is a 2-digit number AC. What is more, the product of the digits of the 2-digit number AC is a 1-digit number C. What number is it?

Problem 26

Five points are put at random on a square whose side length is 2. Show that there will be at least two such points whose distance will not be bigger than $\sqrt{2}$.

Problem 27

There is a group of 80 coins, among which there is one fake coin a little lighter than the others. Having some beam scales, what is the smallest number of weighing to be taken in order to find the fake coin?

Problem 28

There is a group of 12 coins with one fake coin whose weight is different than the others (unknown whether the fake coin is heavier or lighter). With three times of putting on the beam scales, how to determine which coin is fake, and whether it is heavier or lighter than the others?

Problem 29

Two brothers sold a flock of sheep receiving for one sheep as many ducats (gold ancient European currency) as the number of the sheep in the flock. The brothers divided the money they had earned. The older brother took 10 ducats, the younger brother took another 10 ducats, the older took another 10, then the younger took another 10 etc. until the younger brother got the rest of fewer than 10 ducats. How many sheep were there in the flock? How many ducats should the older brother add to the remaining sum in order to make the share just?

2. **Numbers and operations**.

Numbers are the foundation of mathematics. Not only did the Greeks but also other ancient nations start their adventure with numbers, although the Greeks were the ones who left most mathematical discoveries for posterity. There are two branches of mathematics which deal with numbers in a special way; these are arithmetic and the theory of numbers. The problems raised by the latter are the ones which are the most challenging, and some of them were waiting to be solved for a couple of centuries[1]. Nevertheless, let us try to solve some of the less complicated issues.

Problem 30
Let us take a decimal fraction, whose all digits make a sequence of successive natural numbers 0.123456789101112131415... . Show that it is not a recurring decimal.

Problem 31
Let us put successively all natural numbers from 1 to 60. From the number 1234...5960 that has been received, 100 digits are supposed to be crossed out in order to get the smallest possible number. What number is it?

Problem 32
Let us take numbers from 0 to 9999999. Some of them include digit 1, others do not. Which numbers are there more of?

Problem 33
Show that the sum of the digits in number:

$$4444^{4444}$$

is smaller than 159984.

Problem 34
Prove that a 200-digit number, in which there are one hundred 1s at the beginning, and then one hundred 2s is a product of natural numbers.

Problem 35
Find the sum of all digits of natural numbers from 1 to 1000000.

Problem 36
All digits of number 2^{1971} have been written, and just after them all digits of number 5^{1971}. How many digits have been written altogether?

[1] The famous Fermat's theorem was proved only in 1993 r. (by Andrew John Wiles) and it had been published in 1670 r. (after the author's - Pierre Fermat's - death). The discussion whether the theorem was proved correctly took two years.